

Shear viscosity and spin diffusion coefficient of a two-dimensional Fermi gas

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Using kinetic theory, we calculate the shear viscosity and the spin diffusion coefficient as well as the associated relaxation times for a two-component Fermi gas in two dimensions, as a function of temperature, coupling strength, polarization, and mass ratio of the two components. It is demonstrated that the minimum value of the viscosity decreases with the mass ratio, since Fermi blocking becomes less efficient. We furthermore analyze recent experimental results for the quadrupole mode of a 2D gas in terms of viscous damping obtaining a qualitative agreement using no fitting parameters.

INTRODUCTION

The properties of 2-dimensional (2D) Fermi systems are fundamental for our understanding of a wide range of phenomena including organic and high- T_c superconductors, 2D nano-structures, and ^3He films. A new generation of experiments are now probing the many-body properties of atomic Fermi gases in 2D traps [1–3]. This provides a unique possibility to systematically explore the physics of 2D systems using the high experimental control characterizing atomic gases. Recently, there has been a lot of interest in the transport properties of atomic gases. One reason is that transport coefficients provide excellent probes for strong correlations, since they can change by orders of magnitude due to interactions. It has been shown experimentally that 3D atomic gases may form a perfect fluid with a shear viscosity η having the least possible value consistent with quantum mechanics [4]. This has inspired a lot work investigating the connections between the physics of atomic gases, and other strong coupling systems including quark-gluon plasmas, and liquid Helium [5]. Also, recent experiments demonstrate that the spin diffusion coefficient approaches a scale set by quantum mechanics for a 3D resonantly interacting atomic gas [6]. The first experiments probing the collective mode spectrum of a strongly interacting 2D Fermi gas were recently reported [7]. The frequency of the breathing mode was shown to be provide evidence of a classical dynamical scaling symmetry [8], whereas the damping of the quadrupole mode was used as a measure for the shear viscosity of a 2D Fermi gas.

We calculate the shear viscosity and the spin diffusion coefficient as well as the associated relaxation times for a two-component Fermi gas in 2D using kinetic theory. The dependence of the viscosity on the mass ratio of the two components is analyzed, and we show that the minimum value is reduced for systems with a mass imbalance. We furthermore analyze the recent experimental results for the quadrupole mode in terms of viscous damping [7] obtaining a qualitative agreement. However, our analysis shows that further work is needed to understand the experiments quantitatively.

FORMALISM

Consider a 2D gas of two fermionic species $\sigma = 1, 2$ with mass m_σ and density $n_\sigma = k_{F\sigma}^2/4\pi$ so that the total density is $n = n_1 + n_2$. The range of the interaction is taken to be much shorter than the interparticle spacing, and there is therefore no interaction between identical fermions. We shall focus on two steady state non-equilibrium situations: one with a spatially varying local mean velocity $\mathbf{u}(\mathbf{r})$, and one with a spatially varying magnetization $M(\mathbf{r}) = n_1(\mathbf{r}) - n_2(\mathbf{r})$, which we for concreteness take to have the forms $\mathbf{u}(\mathbf{r}) = [u_x(y), 0]$ and $M(\mathbf{r}) = M(x)$. As a result of the velocity field $u_x(y)$, there is a net current Π_{xy} in the y -direction of momentum along the x -direction, and likewise $M(x)$ induces a net magnetization current j_M in the x -direction. Within linear response, we can write

$$\Pi_{xy} = -\eta \partial_y u_x \quad \text{and} \quad j_M = -D \partial_x M \quad (1)$$

which defines the shear viscosity η and the spin diffusion coefficient D .

We briefly outline a variational method to calculate the shear viscosity and the spin diffusion coefficient within kinetic theory. Further details are given in Refs. [9–12]. In kinetic theory, both coefficients are obtained from a steady state solution to the Boltzmann equation. In the hydrodynamic limit, the distribution functions $f_\sigma(\mathbf{r}, \mathbf{p})$ are close to the local equilibrium form $f_\sigma^{\text{le}} = 1/[\exp(\beta \xi_\sigma^{\text{le}}) + 1]$ with $\xi_\sigma^{\text{le}} = p^2/2m_\sigma - \mathbf{u}(\mathbf{r}) \cdot \mathbf{p} - \mu_\sigma$ for the case of a local velocity field \mathbf{u} and $\xi_\sigma^{\text{le}} = p^2/2m_\sigma - \mu_\sigma(\mathbf{r})$ appropriate for a local magnetization M . Here $\mu_1(\mathbf{r})$ and $\mu_2(\mathbf{r})$ are the spatially varying chemical potentials corresponding to the magnetization and $\beta = 1/T$ (we use units where $k_B = \hbar = 1$). When these local equilibrium functions are plugged into the left side of the linearized Boltzmann equation, it can be written as

$$\frac{\partial f_\sigma^0}{\partial \epsilon} \Phi_\sigma^\eta \frac{\partial u_x}{\partial y} = I_\sigma \quad \text{and} \quad \frac{\partial f_\sigma^0}{\partial \epsilon} \Phi_\sigma^D \frac{\partial \mu_\sigma}{\partial x} = I_\sigma \quad (2)$$

where I_σ is the collision operator for component σ , and $f_\sigma^0 = 1/[\exp(\beta(\epsilon_\sigma - \mu_\sigma)) + 1]$ is the equilibrium function with $\epsilon_\sigma = p^2/2m_\sigma$. Here $\Phi_\sigma^\eta = p_x p_y / m_\sigma$ and $\Phi_\sigma^D = p_x / m_\sigma$ for shear viscosity and spin diffusion respectively.

The momentum and spin currents are given by

$$\begin{aligned}\Pi_{xy} &= \int d^2\tilde{k} (\Phi_1^\eta f_1 + \Phi_2^\eta f_2) \\ j_M &= \int d^2\tilde{k} (\Phi_1^D f_1 - \Phi_2^D f_2)\end{aligned}\quad (3)$$

with $d^2\tilde{k} = d^2k/(2\pi)^2$. To proceed, we need an approximate solution to the Boltzmann equation (2). In 3D, the ansatz $\delta f_\sigma \propto \Phi_\sigma f_\sigma^0(1 - f_\sigma^0)$ for the deviation of $f_\sigma(\mathbf{r}, \mathbf{p})$ away from equilibrium is known to yield results within 2% of the exact result for the viscosity [9, 13]. We therefore use this ansatz for the 2D case which yields

$$\eta = 2\beta \frac{\langle \Phi^\eta \rangle^2}{\langle \Phi^\eta H[\Phi^\eta] \rangle} \quad \text{and} \quad D = \frac{\beta}{\chi} \frac{\langle \Phi^D \rangle^2}{\langle \Phi^D H[\Phi^D] \rangle}. \quad (4)$$

as variational expressions for the viscosity and spin diffusion coefficient. We have defined the average $\langle \Phi^2 \rangle \equiv 2^{-1} \sum_\sigma \int d^2\tilde{k} f_\sigma^0(1 - f_\sigma^0) \Phi_\sigma^2$ and $\chi = \partial(n_1 - n_2)/\partial(\mu_1 - \mu_2)$ is the magnetic susceptibility. The linearized collision integral can after symmetrization be written as

$$\begin{aligned}\langle \Phi H[\Phi] \rangle &= \frac{1}{4} \int d^2\tilde{k}_1 d^2\tilde{k}_2 \frac{p_r}{m_r} \int_0^{2\pi} d\theta' \frac{d\sigma}{d\theta'} \\ &\times (\Delta\Phi)^2 f_1^0 f_2^0 (1 - f_1^0)(1 - f_2^0)\end{aligned}\quad (5)$$

where $\mathbf{p}_r = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/M$ is the relative momentum of the incoming scattering particles, θ' is the angle between the outgoing and incoming relative momenta, and $d\sigma/d\theta$ is the differential cross section. The total mass is $M = m_1 + m_2$, and $m_r^{-1} = m_1^{-1} + m_2^{-1}$ is the reduced mass. We have defined the function $\Delta\Phi \equiv \Phi_1^\eta + \Phi_2^\eta - \Phi_3^\eta - \Phi_4^\eta$ for shear viscosity and $\Delta\Phi \equiv \Phi_1^D - \Phi_2^D - \Phi_3^D + \Phi_4^D$ for spin diffusion. It determines the contribution of a given collision to the momentum and spin transport respectively. The reason for the factor 2 difference in the expressions for η and D in (4) is that the two components contribute with the same sign to the momentum current and with opposite signs to the magnetic current, see (3). This also causes the sign differences in the expressions for $\Delta\Phi$ in the two cases.

Relaxation times

The viscous and spin relaxation times τ_η and τ_D which give the typical time between collisions for the two types of motion are useful for estimating whether a system is in the hydrodynamic regime. Suitable definitions can be obtained by writing the collision integral as $I_\sigma \simeq \delta f_\sigma/\tau$, which gives $\eta = 2\tau_\eta\beta\langle(\Phi^\eta)^2\rangle$ and $D = \tau_D\beta\langle(\Phi^D)^2\rangle/\chi$. Performing the integrals yields

$$\frac{\eta}{n} = \tau_\eta \frac{\int_0^\infty d\epsilon \epsilon f^0}{\int_0^\infty d\epsilon f^0} \quad \text{and} \quad \frac{D}{n} = \frac{\tau_D}{2m\chi} \quad (6)$$

where we have taken $m_1 = m_2$ and $n_1 = n_2 = n/2$ for simplicity. This reduces to $\eta = n\tau_\eta\epsilon_F/2$ and $D = \epsilon_F\tau_D/m$ in the degenerate limit, whereas $\eta = n\tau_\eta T$ and $D = \tau_D T/m$ in the classical limit. We have used $\chi = m/2\pi$ for $T \ll T_F$ and $\chi = n/2T$ for $T \gg T_F$.

Scattering cross section

When the range of the interaction is much shorter than the typical interparticle spacing, the scattering between the $\sigma = 1$ and $\sigma = 2$ fermions is predominantly s -wave. The 2D cross section for relative momentum p_r is $\sigma = m_r^2 |\mathcal{T}(p_r^2/2m_r)|^2/p_r$ with the \mathcal{T} -matrix given by [14–16]

$$\mathcal{T}(\epsilon) = \frac{2\pi}{m_r} \frac{1}{\ln(|E_b|/\epsilon) + i\pi} \quad (7)$$

which has a pole at a 2-body bound state with energy $E_b = -1/2m_r a_2^2$.

CLASSICAL LIMIT

Consider the classical limit $T \gg T_{F\sigma} = k_{F\sigma}^2/2m_\sigma$. In this limit $f_\sigma^0 \ll 1$, and the integrals in (4) are straightforward to perform. We obtain for the viscosity

$$\eta_{cl} = \frac{m_r}{2\pi^2} \frac{(n_1 + n_2)^2}{n_1 n_2} \frac{T}{I_\eta(T/E_b)} \quad (8)$$

with

$$\begin{aligned}I_\eta(T/E_b) &= \int_0^\infty dt e^{-t} \frac{t^2}{\ln(|E_b|/Tt)^2 + \pi^2} \\ &\simeq \frac{2}{[\ln(|E_b|/T) - 0.92]^2 + \pi^2}.\end{aligned}\quad (9)$$

The shear viscosity depends only on the reduced mass in the classical limit. This is because the scattering only depends on the relative coordinates in this limit, since there is no Fermi-blocking. For fixed total density $n_1 + n_2$, the viscosity is minimum for $n_1 = n_2$ as expected, since the scattering becomes less frequent with increasing population imbalance. The viscosity in the classical limit for $m_1 = m_2$ and $n_1 = n_2$ was reported while this manuscript was being written [17] and the result agrees with (8)-(9) for that case.

Likewise, (4) yields for the spin diffusion coefficient in the classical limit

$$D_{cl} = \frac{T}{4\pi^2 n} \frac{1}{I_D(T/E_b)} \quad (10)$$

with

$$\begin{aligned}I_D(T/E_b) &= \int_0^\infty dt e^{-t} \frac{t}{\ln(|E_b|/Tt)^2 + \pi^2} \\ &\simeq \frac{1}{[\ln(|E_b|/T) - 0.42]^2 + \pi^2}.\end{aligned}\quad (11)$$

Here, we have for simplicity taken $m_1 = m_2$ and $n_1 = n_2 = n/2$ and used $\chi = n/2k_B T$ in the classical limit.

Equations (8)-(9) and (10)-(11) should be compared with the analogous expressions obtained for the 3D case: $\eta = 15(mk_B T)^{3/2}/32\sqrt{\pi}$ [11, 18] and $D = 3\sqrt{m}(k_B T)^{3/2}/16\sqrt{\pi}$ [12] for a classical gas in the unitarity regime. The reason for the more complicated T -dependence in 2D is the intrinsic energy dependence of the \mathcal{T} matrix (7), which means one never recovers the simple power law predictions for an energy independent cross section: $\eta \sim nvml_{\text{mf}} \propto \sqrt{T}$ and $D \sim vl_{\text{mf}} \propto \sqrt{T}$ where $l_{\text{mf}} \sim 1/n\sigma$ is the mean free path.

NUMERICAL RESULTS

In this section we present numerical results for the viscosity and spin diffusion coefficient obtained from (4).

Viscosity

In Fig. 1 (a), we plot the viscosity as a function of T for the mass ratios $m_1/m_2 = 1$ and $m_1/m_2 = 6/40$. The latter corresponds to a mixture of ^{40}K and ^6Li atoms. We have taken the density of the two components to be equal and $\ln(|E_b|/T_{F2}) = 1$. For high temperatures, the viscosity approaches the classical result (8), whereas it increases strongly for low T due to Fermi blocking [19]. This results in a minimum of the viscosity at $T \simeq 0.6T_{F2}$ for $m_1/m_2 = 1$, whereas the minimum is located at $T \simeq 1.3T_{F2}$ for $m_1/m_2 = 6/40$ due to the larger Fermi temperature for the light component $\sigma = 1$.

An important result is that the minimum viscosity of the mass imbalanced mixture is significantly smaller than for the mass balanced mixture; for the mass ratio $m_1/m_2 = 6/40$ it is a factor 0.6 smaller. This can be understood as follows: changing both masses keeping $m_1/m_2 = 1$ clearly does not reduce the minimum value of η/n , since this simply amounts to rescaling T_F ; however, reducing m_1 while keeping m_2 fixed makes the Fermi blocking less efficient on the scale of T_{F1} and the minimum value of η/n is reduced essentially since the classical result (8) holds for lower T/T_{F1} . The minimum value of the viscosity is subject to intense interest due to a conjecture inspired by results for a certain class of strong coupling theories [20], which states that the ratio of the viscosity over the entropy of any system obeys the universal bound $\eta/s > 1/4\pi$ [21]. In the inset of Fig. 1 (a), we therefore plot η/s for the same parameters as in the main plot. The entropy density $s = s_1 + s_2$ is obtained from the ideal gas expression $s_\sigma = -\int d^2k [f_\sigma^0 \ln f_\sigma^0 - (1 - f_\sigma^0) \ln(1 - f_\sigma^0)]$. Again, we see that the minimum value of η/s is significantly smaller for the mass ratio $m_1/m_2 = 6/40$. Intriguingly, it seems to follow from kinetic theory that a two-component sys-

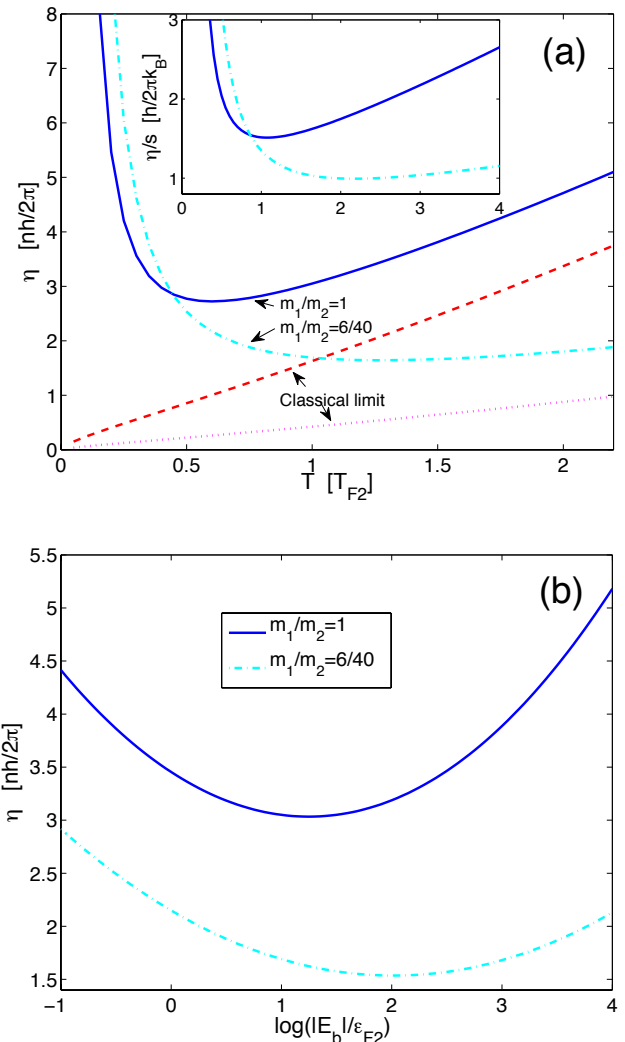


FIG. 1: (color on-line)(a) The viscosity as a function of temperature for $\ln(|E_b|/T_{F2}) = 1$ and $m_1/m_2 = 1$ (blue line), $m_1/m_2 = 6/40$ (cyan dash-dot line). The classical limits are plotted as red dashed and purple dotted lines. The inset shows η/s as a function of T . (b) The viscosity as a function of $\ln(|E_b|/T_{F2})$ for $T = T_{F2}$ and $m_1/m_2 = 1$ (blue line), and $m_1/m_2 = 6/40$ (cyan dash-dot line).

tems with a sufficiently large mass ratio can break the conjectured bound $\eta/s > 1/4\pi$. A similar effect is in fact present for 3D systems.

In Fig. 1 (b), we plot η/n as a function of $\ln(|E_b|/T_{F2})$ for $T = T_{F2}$. The viscosity is minimum in the strong coupling regime $\ln(|E_b|/T_{F2}) \sim \mathcal{O}(1)$ as expected. In the classical limit, it follows from (8) that the minimum is located at $\ln(|E_b|/T) \sim 0.92$. With decreasing m_1/m_2 , the minimum moves to larger values of $\ln(|E_b|/T_{F2})$ because T_{F1} increases.

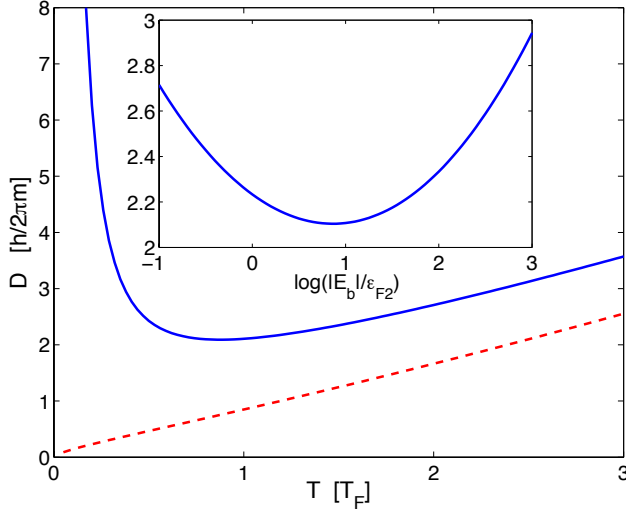


FIG. 2: (color on-line) The spin diffusion coefficient as a function of temperature for $\ln(|E_b|/T_{F2}) = 1$ and $m_1/m_2 = 1$. The classical limit is plotted as a red dashed line. The inset shows D as a function of the interaction strength for $T = T_{F2}$.

Spin diffusion coefficient

The spin diffusion coefficient is plotted in Fig. 2 as a function of temperature for $m_1/m_2 = 1$ and $\ln(|E_b|/T_{F2}) = 1$. For high T , it approaches the classical value (10) whereas Fermi blocking makes it increase strongly for low T , leading to a minimum value at $T = 0.85T_F$. The inset shows D as a function of $\ln(|E_b|/T_{F2})$ for $T = T_{F2}$. Again, the minimum value is for $\ln(|E_b|/T_{F2}) \sim \mathcal{O}(1)$. For high T , (10) predicts $\ln(|E_b|/T) \sim 0.42$ to be the minimum value.

Validity of kinetic theory

Let us briefly discuss the range of validity of kinetic theory. For weak coupling $|\ln(|E_b|/\epsilon_F)| \gg 1$, the kinetic approach is accurate except for extremely low temperature, as has been shown for the viscosity in 3D [18]. For strong coupling, one must expect the Boltzmann equation (2) to break down for low temperature where there are no well-defined quasi-particles. In 3D, calculations of the viscosity based on the Kubo-formalism show that the kinetic approach is accurate down to temperatures significantly below T_F , even for strong coupling [13, 18, 22, 23]. We expect a similar result to hold in 2D. In particular, kinetic theory is likely to be reliable at the temperatures where we predict η and D to be minimum. The occupation of the closed channel molecule can furthermore have significant effects on thermodynamic properties in 2D [24]. Similarly, corrections to the single channel approximation for the \mathcal{T} -matrix (7) could influence the transport properties considered here, although one would

expect small effects for a broad resonance.

EXPERIMENTS

The frequency and damping of the quadrupole mode of a 2D Fermi gas of ^{40}K atoms with equal populations in two hyperfine states were recently measured [7]. The results were interpreted in terms of viscous damping appropriate for the hydrodynamic regime. The amplitude damping of a collective mode can be calculated from [25]

$$\Gamma = \frac{|\langle \dot{E}_{\text{mech}} \rangle_t|}{2\langle E_{\text{mech}} \rangle_t} \quad (12)$$

where $\langle E_{\text{mech}} \rangle_t$ is the time averaged mechanical energy of the mode. Taking the velocity field of the quadrupole mode to have the form $\mathbf{u}(\mathbf{r}) = (x, -y) \cos \omega t$, we get

$$\langle E_{\text{mech}} \rangle_t = \frac{m}{2} \int d^2r n(\mathbf{r}) u^2(\mathbf{r}), \quad (13)$$

where we have used that the potential energy of the mode is equal to the kinetic energy, and we have neglected any interaction energy. The viscous damping is for this velocity field, following [25, 26], given by

$$\langle \dot{E}_{\text{mech}} \rangle = -2b^2 \int d^2r \frac{\eta}{1 + \omega_Q^2 \tau_\eta^2}. \quad (14)$$

Here we have used the real part of the complex dynamical viscosity $\eta(\omega) = \eta/[1 - i\omega\tau_\eta]$ [27] evaluated at the quadrupole frequency ω_Q to obtain a cut-off in the outer classical regions of the cloud, where the viscosity is given by (8) with $m_1 = m_2$ and $n_1 = n_2 = n/2$, and therefore is independent of density. In the classical regime, (12) becomes using (8) and (6)

$$\Gamma_{\text{cl}} = \frac{2\omega}{\pi\sqrt{N}I_\eta} \int_0^\infty du \frac{1}{1 + \omega_Q^2 \tau_\eta^2} \quad (15)$$

where $N = N_1 + N_2$ is the total number of particles trapped and ω is the 2D trapping frequency.

In Fig. 3 (a), we plot the damping of the quadrupole mode taking $T/T_F = 0.47$ and $N = (E_F/\hbar\omega_\perp)^2 = 4300$ particles trapped which are the experimental parameters appropriate for Fig. 1 in Ref. [7]. We have calculated the damping as a function of $\ln(k_F a_2)$ and the \times 's are the experimental results reported in Ref. [7]. We see that the theory agrees qualitatively with the data. In Fig. 3 (b), we plot the damping as a function of T for various coupling strengths taking $N = 3500$ particles trapped to model the experimental situation of Fig. 3 in Ref. [7]. Again, the theory accounts qualitatively for the experimental results which are plotted as \times 's. Note that we have no fitting parameters. For both sets of data, the agreement between theory and experiment is best for

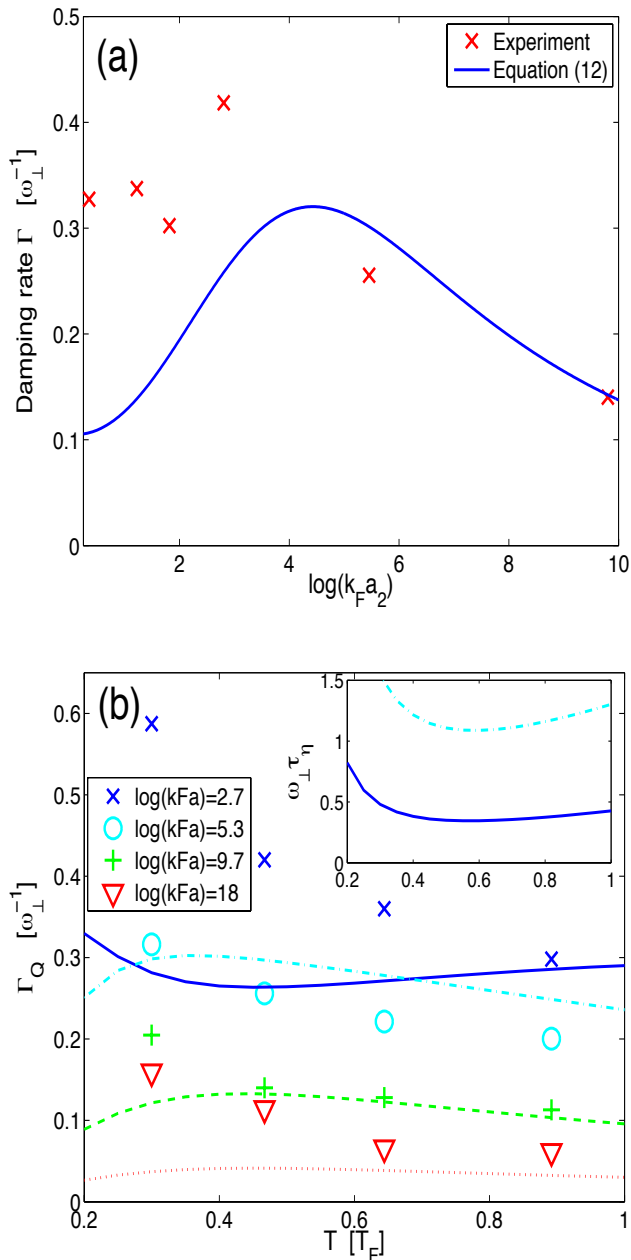


FIG. 3: (color on-line) (a) The damping of the quadrupole mode as a function of interaction strength. The \times 's are the experimental results of Ref. [7]. (b) The damping of the quadrupole mode as a function of T for various coupling strengths with the \times 's the experimental results of Ref. [7]. The inset shows the viscous collision rate $\omega_{\perp} \tau_{\eta}$.

large $\ln(k_F a_2)$ and large T/T_F whereas there are significant quantitative discrepancies in the strong coupling regime of small $\ln(k_F a_2)$. Similar results were reported in Ref. [17] for the spatial average of the viscosity using the classical limit approximation.

It is perhaps surprising that the agreement is best in

the weak coupling regime where the system is collisionless rather than hydrodynamic. This is illustrated in the inset in Fig. 3 (b) which shows $\omega_{\perp} \tau_{\eta}$: the hydrodynamic condition $\omega_{\perp} \tau_{\eta} < 1$ is fulfilled only for $\ln(k_F a_2) = 2.7$, where as $\omega_{\perp} \tau_{\eta} > 1$ for larger $\ln(k_F a_2)$ indicating collisionless dynamics. However, despite being based on hydrodynamics, the viscous damping approach turns out to work rather well in the collisionless regime. In 3D it has in fact been shown to yield exact results in the collisionless limit provided one uses the complex dynamical viscosity evaluated at the collisionless collective mode frequency [26].

The reason for the discrepancy between theory and experiment for small $\ln(k_F a_2)$ where the system is hydrodynamic, can be strong coupling effects making the kinetic approach quantitatively inaccurate as discussed above. Better agreement could also be obtained by solving the Boltzmann equation approximately by taking moments with basis functions for δf_{σ} [23, 26, 28]. Such an approach has indeed been successful in describing the frequency and damping of collective modes in 3D.

To summarize, we have, using kinetic theory, calculated the shear viscosity and the spin diffusion coefficient for a two-component Fermi gas in 2D. Both transport coefficients have a minimum value somewhat below the Fermi temperature. We showed that the minimum value of the viscosity can be reduced significantly with increasing mass ratio of the two components. Using a viscous damping approach, we qualitatively accounted for recent experimental results for the damping of the quadrupole mode of a 2D Fermi gas. However, our results also showed that further work is needed to obtain a quantitative understanding of the results.

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